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A Topological Model for Oersted–Ampère's Law

C. V. WESTENHOLZ

Institut für Angewandte Mathematik der Universität Mainz,
D-6500 Mainz (Germany)

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Abstract

A geometrical description of Oersted–Ampère's law $\oint \mathbf{H} ds = (4\pi/c)I$ can be given in terms of an appropriate topological manifold. More precisely: It will be shown that Oersted–Ampère's law can be related to the topological invariant $H^1(S^1)$, i.e. de Rham's first cohomology group on the differentiable manifold

$$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

1. *Introduction*

A purely geometrical description of certain aspects of classical electrodynamics has already been put forward by Misner & Wheeler (1957). These authors have been led to propose that, within a suitable topological framework, i.e. de Rham's cohomology theory, unquantised (continuously variable) charge receives a natural interpretation in terms of source-free electromagnetic fields that are everywhere subject to Maxwell's equations for free space. That is, charge can be characterised in terms of lines of force which are trapped in a multiply connected topological space M. This charge model can be summarised as follows: Let

$$\omega = -(H_i\, dx^i) . dx^0 + E_1\, dx^2\, dx^3 + E_2\, dx^3\, dx^1 + E_3\, dx^1\, dx^2 \qquad (1.1)$$

(where $\{E = (E_i), H = (H_i)\}$ denotes the electromagnetic field) be a closed 2-form on M^4. If c^2 is a closed surface in the $x^0 =$ const. hyperplane of the space-time M^4, charge is defined by de Rham's period

$$/\int_{c^2} \omega = \int {}^*E_{ij}\, dx^i\, dx^j = 4\pi e \qquad (1.2)$$

where $/E_1 = {}^*E_{23}$, $/E_2 = {}^*E_{31}$, $/E_3 = {}^*E_{12}$. This equation can be given a straight-forward generalisation to an empty curved space time M^4, which may serve as a model for charge, provided its second Betti-number be $\beta_2 \geqslant 1$ and

$$\omega \in \mathring{F}(M^4) = \{\omega \in F^2 : d\omega = 0\} \qquad (1.3)$$

(vector space of closed 2-forms, d denotes the exterior derivative). The form (1.1) therefore provides a charge reinterpretation in terms of Maxwell's equations

$$d\omega = 0 \tag{1.4}$$

which comprises the relevant equation $\operatorname{div} \mathbf{E} = 0$.

In this paper we are concerned with the problem of defining another type of field which is derived from geometry. We examine its properties by referring to the case of electromagnetism as exhibited in Misner & Wheeler (1957). We therefore wish to express first the classical Maxwell field $F_{\mu\nu}$ as a field derived from the properties of a curved empty multiply connected topological space M^4, such that the flux (1.2) through the 'holes' of this topology remains invariant. Otherwise stated, this amounts to re-expressing the field (1.1) in a purely geometrical form such that it entails the existence of the constant of motion which represents the charge e. This can be done as follows:

Define this topological Maxwell field by means of the pairing

$$(\omega, c_i^2), \qquad \text{where} \qquad \omega = \sum F_{\mu\nu}\, dx^\mu . dx^\nu \in \mathring{F}^2(M^4)$$

and

$$c_i^2 \in \mathring{C}_2(M^4) = \{c \colon \partial c = 0\} \tag{1.5}$$

the group of 2-cycles on M^4. ∂: linear boundary operator ($i = 1 \dots \beta_2$ denotes the second Betti number).

Clearly the field (1.5) is characterised by means of the topological invariants $H_2(M^4)$ and $H^2(M^4)$, the homology and cohomology groups of M^4 respectively (von Westenholz, 1972, and preprint). In such a model it turns out that the charge e accounts for the physical interpretation of these topological invariants. More precisely, there exists a bilinear mapping

$$H^2 \times H_2 \to \mathbb{R}$$
$$(\omega, c) \to \int_c \omega = 4\pi e \tag{1.6}$$

which yields the required interpretation.

Remark 1: By virtue of de Rham's first theorem, the bilinear mapping (1.6) is non-degenerate and therefore establishes a duality relationship between H^2 and H_2.

Remark 2: The homotopy groups $\prod_k (M^n)$, $k = 1, 2 \dots$ represent topological invariants as well and therefore can provide additional geometric conditions for fields which are described in terms of some geometry M^n. A conservative force field, $\mathbf{F} = -\operatorname{grad} \varphi$ ($\omega = \sum_i F_i\, dx^i = -d\varphi$), for instance, may be characterised in terms of the fundamental group $\prod_1 (M^n)$ and thus presents an interesting problem in this respect (von Westenholz, 1972).

The aim of this paper is to express Oersted–Ampère's law in terms of some appropriate topological model, i.e. some empty curved space.

2. *Oersted–Ampere's Law as Property of Some Curved Empty Space*

Steady-state magnetic phenomena are characterised by the basic law (2.1). That is, consider an infinitely long straight current-carrying wire $\{(x,y,z): x^2+y^2=a^2, z \text{ unrestricted}\}$. The lines of magnetic field are concentric circles c^1 around this wire and Oersted–Ampère's law can be written in the form

$$\oint_{c^1} \mathbf{H}.d\mathbf{s} = \iint_{c^2} \operatorname{rot}\mathbf{H}.d\mathbf{f} = \frac{4\pi I}{c} \tag{2.1}$$

$I = \iint \mathbf{i}.d\mathbf{f}$ denotes the total current ($\mathbf{i}$: current density). The value of this line-integral is independent of the choice of the circle c^1 and is given outside the wire (where $|\mathbf{i}| = 0$), by

$$H = \frac{2I}{cr} \tag{2.2}$$

(where r denotes the radius of c^1, $r > a$). The source-free basic differential law of magnetostatics,

$$\operatorname{rot}\mathbf{H} = 0 \tag{2.3}$$

yields

$$H_x = \frac{-2I}{c}\frac{y}{x^2+y^2}, \qquad H_y = \frac{2I}{c}\frac{x}{x^2+y^2}, \qquad H_z = 0 \tag{2.4}$$

With regard to the law inside the wire, refer to Remark 9 below.

Now introduce the 1-form

$$\omega = \frac{2I}{c}\left[\frac{-y.dx}{x^2+y^2} + \frac{x.dy}{x^2+y^2}\right] \in F^1(\mathbb{R}^2 - \{0\})\dagger \tag{2.5}$$

which is readily checked to be closed on $\mathbb{R}^2 - \{0\}$. The form (2.5) stands for the magnetic field (2.2) or (2.4), more precisely:

$$\omega := \mathbf{H}.d\mathbf{s} \tag{2.6}$$

where $d\mathbf{s} = (dx, dy)$. For later convenience we call (2.5) the *physical* Oersted–Ampère field, contrary to the *geometrical* Oersted–Ampère field (equation (2.12)) which is of the type (1.5). Consequently, equation (2.3) reads:

$$d\omega = 0 \tag{2.7}$$

The corresponding basic law with source $\mathbf{i}$ is given by

$$d\tilde{\omega} = \frac{4\pi}{c}\gamma, \qquad \gamma \in F^2(M) \qquad \left(\operatorname{rot}\mathbf{H} = \frac{4\pi\,\mathbf{i}}{c}\right) \tag{2.8}$$

† It is immaterial to consider the domain $\mathbb{R}^2 - \{0\}$ instead of $\mathbb{R}^2 - D_r$ (D_r denotes the disk of radius $0 < r \leq a$), since these spaces are topologically equivalent (refer to our subsequent Remark 9). (We have dropped the z-direction along the axis of the wire since the physical space of interest is the plane minus a disk.)

Now we can assert

Lemma 1: Let ω be given by formula (2.5), then the following statements are equivalent:

(a) The de Rham periods $\int_{c_i} \omega$ depend only upon the homology class $\{c\} \in H_1(\mathbb{R}^2 \backslash \{0\})$ ($H_1(\mathbb{R}^2 \backslash \{0\})$ denotes the first homology group on $\mathbb{R}^2 \backslash \{0\}$).

(b) Oersted–Ampère's law (2.1) is independent of the choice of the line of magnetic field.

Proof: The lines of magnetic field are cycles on $\mathbb{R}^2 \backslash \{0\}$, which are all homologous to each other, as seen in the adjacent Fig. 1. In fact, statement (a) means:

$$c_i{}^1 \sim c_j{}^1 \Leftrightarrow c_i{}^1 - c_j{}^1 = \partial c_{ij}^2 \Rightarrow \int_{\partial c_{ij}^2} \omega = \int_{c_i{}^1 - c_j{}^1} \omega = \int_{c_i{}^1} \omega - \int_{c_j{}^1} \omega = 0$$

(definition of homologous cycles).

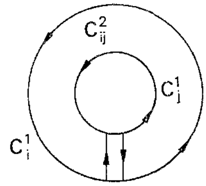

Figure 1.

Since by Stoke's theorem: $\int_{\partial c} \omega = \int d\omega = 0$ for the closed form (2.5), one obtains

$$\frac{4\pi I}{c} = \int_{c_1{}^1} \omega = \int_{c_2{}^1} \omega = \ldots = \int_{c_u{}^1} \omega \tag{2.9}$$

which is just formula (2.1). Conversely, all lines of magnetic field are elements of the same homology class $\{c^1\} \in H_1(\mathbb{R}^2 \backslash \{0\})$, thus obviously statement (a) holds.

A mathematical model which accounts for the physical law (2.2) or (2.4), respectively, is provided by the differentiable manifold

$$S^1 = \{(x, y) \in \mathbb{R}^2 / x^2 + y^2 = 1\}$$

which is related to the physical configuration space $\mathbb{R}^3$ by means of any smooth map

$$\psi : S^1 \subset \mathbb{R}^2 \to \mathbb{R}^3 \tag{2.10}$$

Then the induced mapping

$$\psi^* : \mathring{F}^1(\mathbb{R}^2 \backslash \{0\}) \to \mathring{F}^1(S^1) \tag{2.11}$$

$$\omega = \frac{2I}{c}\left[\frac{-y \, . \, dx + x \, . \, dy}{x^2 + y^2}\right] \to \tilde{\omega} = f(\vartheta) \, . \, d\vartheta$$

denotes the pull-back map on the closed differential forms

$$\mathring{F}^1 = \{\omega \in F^1 / d\omega = 0,\ d\colon F^1 \to F^2\}$$

Consequently, define a topological Oersted–Ampère field by means of the pairing

$$(\tilde{\omega}, \tilde{c}_i^{\,1}); \qquad \psi^* \omega = \tilde{\omega} \in \mathring{F}^1(S^1), \qquad \tilde{c}_i^{\,1} \in \mathring{C}_1(S^1), \qquad i = 1, 2 \ldots \quad (2.12)$$

which constitutes a geometrical version of the physical Oersted–Ampère field (2.5).

Remark 3: Definition (2.12) is compatible with Lemma 1 and formula (2.9) by virtue of the definition

$$\int_{\tilde{c}} \tilde{\omega} = \int_{\tilde{c}} \psi^* \omega \overset{\text{def}}{=} \int_{\psi_* \tilde{c} = c} \omega \quad (2.13)$$

ψ_* is the modul homomorphism, which is induced by the map (2.10) and which maps cycles on S^1 into cycles on $\mathbb{R}^2 \backslash \{0\}$ according to

$$\psi_*\colon \mathring{C}_1(S^1) \to \mathring{C}_1(\mathbb{R}^2 \backslash \{0\}) \quad (2.14)$$

In connection with (2.12) it must be stressed that the first homology group of S^1 is given by

$$H_1(S^1) = \mathring{C}_1(S^1) = \mathbb{Z} \quad (2.15)$$

(group of integers), that is: $H_1(S^1) = \{\ldots -2c^1, -c^1, 0, c^1 \ldots\}$ where nc^1 winds n times around S^1. Therefore no two cycles of S^1 are homologous to each other.

Remark 4: The first Betti number of $\mathbb{R}^2 - \{0\}$ is $\beta_1 = 1$, and the closed 1-form $\overset{1}{\omega} = 2I/c(x.dy - y.dx)/r^2$ is the only independent one.

A topological model for Oersted–Ampère's law can now be exhibited in terms of the following

> *Theorem*: Oersted–Ampère's law can be expressed in terms of de Rham's first cohomology group $H^1(S^1)$ on S^1.

Before starting the proof, we proceed to the following analysis.

(1) Suppose the vanishing of de Rham's period (2.9): $\int_{c_i} \omega = 0$, $d\omega = 0$ then

$$\omega = -d\varphi_m \quad (2.16)$$

i.e. ω is exact. That is, one is faced with the physical situation that $\text{rot}\,\mathbf{H} = 0$ permits the expression of the vector $\mathbf{H}$ as gradient of a magnetic scalar potential φ_m:

$$\mathbf{H} = -\text{grad}\,\varphi_m \quad (2.17)$$

which is equivalent to (2.16). The region of interest is characterised by the following

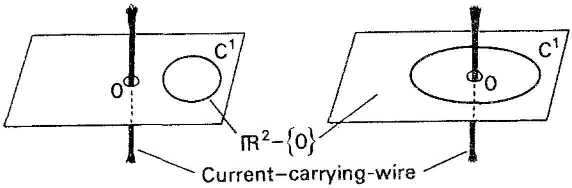

Figure 2 Figure 3

In the case of Fig. 2 we have

$$\int_{c^1} \omega = \oint \mathbf{H}\, d\mathbf{s} = 0$$

(2) The physical situation of formulae (2.1) and (2.9) is displayed in Fig. 3 and we have $\int_{c^1} \omega \neq 0$; $\omega \neq -d\varphi_m$ which invalidates (2.17), i.e. ω is not exact any more.† This lack of exactness of ω on $\mathbb{R}^2 - \{0\}$ can be remedied, however, through the following

Lemma 2: The 1-form

$$\tilde{\omega} - a.d\vartheta, \qquad a \in \mathbb{R}, \qquad \tilde{\omega} = \psi^* \omega \tag{2.18}$$

is exact on S^1.

Remark 5: Since ψ^* maps exact forms into exact forms, we shall use, by abuse of language, the same notation for $d\varphi_m$ on both spaces S^1 and $\mathbb{R}^2\backslash\{0\}$.

The proof of Lemma 2 is based upon

Lemma 3: A 1-form $\tilde{\omega}' \in F^1(S^1)$ is exact if and only if its de Rham period vanishes:

$$\int_{S^1} \tilde{\omega}' = 0 \tag{2.19}$$

Proof: Condition (2.19) is necessary. In fact, let $\tilde{\omega}' = f(\vartheta)\, d\vartheta, f(\vartheta + 2\pi n) = f(\vartheta)$ for every integer $n \in \mathbb{Z}$. $\tilde{\omega}'$ is exact, i.e. $\tilde{\omega}' \in dF^0$ if and only if there exists a periodic function g such that $dg = f(\vartheta)\, d\vartheta$. Then

$$\int_0^{2\pi} \frac{dg}{d\vartheta} d\vartheta = g(2\pi) - g(0) = 0 = \int_{S^1} \tilde{\omega}'$$

The sufficiency of (2.19) is obvious. Set

$$g(\vartheta) = \int_0^{\vartheta} f(\vartheta')\, d\vartheta' \bmod 2\pi$$

† The form $\omega = 2I/c(x.dy - y.dx)/r^2 = 2I/c.d[\operatorname{arc} tg(y/x)]$ is *not* exact, i.e. $\operatorname{arc} tg(y/x) \notin F^0(\mathbb{R}^2\backslash\{0\})$ since $\operatorname{arc} tg(y/x)$ is not continuous on cycles $c^1 \in \mathring{C}_1(\mathbb{R}^2\backslash\{0\})$.

then

$$\frac{dg}{d\vartheta} = f(\vartheta) \Rightarrow dg = \tilde{\omega}'$$

Proof of Lemma 2: Let

$$a = \frac{1}{2\pi} \int_{S^1} \tilde{\omega}$$

$$\int_0^{2\pi} d\vartheta = 2\pi \Rightarrow \int_{S^1} \tilde{\omega} - a\, d\vartheta = 0 \tag{2.20}$$

therefore $\tilde{\omega} - a d\vartheta$ is exact, which achieves the proof.

Since any 1-form on S^1 is closed, Lemma 2 states that every 1-form on S^1 differs from a real multiple of $d\vartheta$ by an exact form, that is $\tilde{\omega} = a.d\vartheta - d\varphi$. We therefore set

$$\tilde{\omega} - a.d\vartheta = -d\varphi \tag{2.21}$$

That is, if

$$\begin{aligned} \tilde{\omega}_1 &= a_1\, d\vartheta - d\varphi_1 \\ \tilde{\omega}_2 &= a_2\, d\vartheta - d\varphi_2 \end{aligned} \Rightarrow \tilde{\omega}_1 - \tilde{\omega}_2 = (a_1 - a_2)\, d\vartheta - (d\varphi_1 - d\varphi_2)$$

i.e. $\tilde{\omega}_1 \nsim \tilde{\omega}_2$ ('$\nsim$' means that $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are not homologous to each other since, by (2.20), $d\vartheta$ is not exact).

Lemma 4: Let $\tilde{\omega}_i$ and $\tilde{\omega}_j$ be representatives of any two cohomology classes of $H^1(S^1)$, then

$$\tilde{\omega}_i = \lambda_{ij}\, \tilde{\omega}_j \qquad \text{for some } \lambda_{ij} \neq 1 \tag{2.22}$$

Conversely, (2.22) entails $\tilde{\omega}_i \nsim \tilde{\omega}_j$.

Remark 6: A system of representatives of

$$H^1(S^1) = \mathring{F}^1/dF^0 = \{dF^0, a_1\, d\vartheta + dF^0, a_2\, d\vartheta + dF^0, \ldots\}$$

will be designated by the isomorphic set $\{0, a_1 d\vartheta, a_2 d\vartheta \ldots\}$.

Proof: Let $\tilde{\omega}_i = a_i d\vartheta$ and $\tilde{\omega}_j = a_j d\vartheta$, a_i and $a_j \in \mathbb{R}$ be representatives of different cohomology classes. This yields

$$\tilde{\omega}_i = \lambda_{ij}\, \tilde{\omega}_j \qquad \text{where} \qquad \lambda_{ij} = \frac{a_i}{a_j}$$

Conversely, let

$$\tilde{\omega}_i = \lambda_{ij}\, \tilde{\omega}_j \qquad \lambda_{ij} \neq 1 \Rightarrow \tilde{\omega}_j - \tilde{\omega}_i \equiv -(1 - \lambda_{ij})\, \tilde{\omega}_i \notin dF^0$$

therefore $\tilde{\omega}_i \nsim \tilde{\omega}_j$.

Physical Interpretation of Lemma 4

$$\omega_i \triangleq H(r_i) = \frac{2I}{cr_i} \qquad H(r_j) = \frac{r_i}{r_j} H(r_i) = \lambda_{ij} H(r_i) \tag{2.22'}$$

$$\omega_j \triangleq H(r_j) = \frac{2I}{cr_j}$$

That is, upon the identification $a_i = r_i$ and $a_j = r_j$ the last formula corresponds to (2.22).

This physical interpretation can be seen to be consistent with Oersted–Ampère's law since the following holds.

Corollary. The non-cohomologous forms $\tilde{\omega}_j$ and $\tilde{\omega}_i = \lambda_{ij}\tilde{\omega}_j$ on S^1 satisfy the condition

$$\int_{\tilde{c}_i} \tilde{\omega}_i = \int_{\tilde{c}_j} \tilde{\omega}_j = \frac{4\pi I}{c} \qquad \text{(for some } \tilde{c}_i \text{ and } \tilde{c}_j \text{ on } S^1\text{)} \tag{2.23}$$

i.e. Oersted–Ampère's law.

Statement (2.23) must hold in any event. It accounts for the correct choice of the topological model, since (2.22) corresponds to (2.22′). In fact:

$$\frac{4\pi I}{c} = \int_{\tilde{c}_i} \tilde{\omega}_i$$

and using (2.22) one has

$$\int_{\tilde{c}_i} \tilde{\omega}_i = \lambda_{ij} \int_{\tilde{c}_j} \tilde{\omega}_j = \int_{\lambda_{ij}\tilde{c}_j} \tilde{\omega}_j = \int_{\tilde{c}_j} \tilde{\omega}_j$$

where $\tilde{c}_j := \lambda_{ij}\tilde{c}_i$ (with the definition $\int_c \omega = \sum a_i \int_{\sigma_i} \omega$ where $c = \sum a_i \sigma_i$, $\sigma_i = \{\varphi : s_i \subset R^m \to M^n\}$ denotes a general simplex (s_i: Euclidean simplex).

Remark 7: The above-mentioned interpretation makes use of the linearity of ψ^*:

$$\omega_i = \lambda_{ij}\,\omega_j \Rightarrow \tilde{\omega}_i = \psi^*\omega_i = \psi^*(\lambda_{ij}\,\omega_j) = \lambda_{ij}\,\psi^*\omega_j = \lambda_{ij}\,\tilde{\omega}_j$$

Remark 8: If $\lambda \in \mathbb{Z}$, $\tilde{c}_j$ is a cycle which satisfies $\forall \tilde{c}_i : \tilde{c}_i \nsim \tilde{c}_j$, since $H_1(S^1) \cong \mathbb{Z}$. If $\lambda \in R$, $\lambda \notin \mathbb{Z}$, $\tilde{c}_j$ is a chain of S^1 and the expression

$$\int_{\tilde{c}_j} \tilde{\omega}_j$$

is no de Rham period any longer.

Remark 9: Oersted–Ampère's law is also valid inside the wire of radius $r > 0$ (cf. Fig. 4).

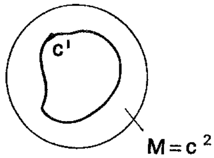

Figure 4.

$$\int_{c^1} \omega = \int_{\partial c^2} \omega = \int_{c^2} d\omega = \frac{4\pi}{c} \int_{c^2} \gamma \tag{2.24}$$

where $c^1 \sim 0$, i.e. c^1 is homotopic to 0, i.e. $c^1 \in \prod_1 (M)$ (Poincaré group of M) and $\prod_1 (M) = 0$. The basic law is (2.8): $d\tilde{\omega} = (4\pi/c)\gamma$, where $\gamma \in F^2(M)$ represents the current density **i** and ω is obviously not closed any longer.

3. *Discussion*

Since

$$H^1(S^1) = \mathring{F}^1/dF^0(S^1) \cong \{a.\ d\vartheta : a \in \mathbb{R}\} \tag{3.1}$$

the interpretation of Oersted–Ampère's law in terms of elements of $H^1(S^1)$ is related to the following two cases:

(1) $\tilde{\omega} = -d\varphi_m$ ($\mathbf{H} = -\text{grad}\,\varphi_m$) (Fig. 2), is associated with the space of exact 1-forms on S^1, i.e.

$$dF^0(S^1) = \{d\varphi : \varphi \in C^\infty (S^1, \mathbb{R})\}$$

that is, the identity element of $H^1(S^1)$. This case corresponds to $a = 0$.

(2) $a \neq 0$, consequently $\omega \notin dF^0(\mathbb{R}^2\backslash\{0\})$ (Fig. 3). That is $\omega \neq -d\varphi_m$ and Oersted–Ampère's law must be interpreted in terms of elements other than the identity element of $H^1(S^1)$.

The cohomology and homology groups $H^1(S^1)$ and $H_1(S^1)$ are topological invariants. Their physical interpretation is given in terms of the Ampère current I which is related to these invariants by means of the bilinear map

$$H^1 \times H_1 \to \mathbb{R}$$

$$(\tilde{\omega}, \tilde{c}) \to \int_{\tilde{c}} \tilde{\omega} = \int_{c} \omega = \frac{4\pi}{c} I \tag{3.2}$$

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